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Asymptotic expansions of the generalized Bessel polynomials[1]

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Abstract

In this paper, we investigate the asymptotic behavior of the generalized Bessel polynomials $y_n(z;a)$. Let $z=\alpha/(n+1)$. We first derive infinite asymptotic expansions for $y_n(z;a)$ when α lies in various regions of the complex plane, except when α is near $\pm\mathrm{i}$. Then we construct uniform asymptotic expansions for $y_n(z;a)$ in neighborhoods of $\alpha=\pm\mathrm{i}$. These expansions involve the Airy function and its derivative. Finally, we use these approximations to study the asymptotic behavior of the zeros of $y_n(z;a)$ near $\alpha=\mathrm{i}$.

Keywords: Generalized Bessel polynomials; Steepest descent method; Uniform asymptotic expansions; Zeros

AMS classification: Primary 41A60; 33C45

1. Introduction

Generalized Bessel polynomials are defined explicitly by

$$y_n(z;a)=\sum_{k=0}^{n}\binom{n}{k}(n+a-1)_k\left(\frac{z}{2}\right)^k,$$

where a is a real number, $(\alpha)_0=1$ and $(\alpha)_k=\alpha(\alpha+1)\cdots(\alpha+k-1)$ for $k\geqslant 1$. In hypergeometric notation, we have

$$y_n(z;a)={}_2F_0(-n,a+n-1;-\tfrac{1}{2}x).$$

These polynomials satisfy the second order linear differential equation

$$z^2y''+(az+2)y'-n(n+a-1)y=0,$$

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and have the integral representation

$$y_n(z;a) = \frac{1}{\Gamma(n-a+1)} \int_0^\infty u^{n+a-2} \left(1 + \frac{uz}{2}\right)^n \mathrm{e}^{-u} \mathrm{d}u. \tag{1.1}$$

A generating function for these polynomials is given by

$$\begin{aligned}(1-2tz)^{-1/2} &\left\{\frac{1}{2}[1+(1-2tz)^{1/2}]\right\}^{2-a} \exp\left\{\frac{1-(1-2tz)^{1/2}}{z}\right\} \\ &= \sum_{k=0}^{\infty} \frac{y_k(z;a)}{k!} t^k. \end{aligned} \tag{1.2}$$

The importance of these polynomials is realized in (1949) when Krall and Frink [6] found its connection with the wave equation in spherical coordinates. At about the same time, Thomson [10] also independently discovered these polynomials in his study of electrical networks. For a historical survey and discussion of many interesting properties, we refer to the definitive book by Grosswald [5].

In the present paper, we are only concerned with the asymptotics of these polynomials. In 1962, Docěv proved that

$$y_n(z;a) = \left(\frac{2nz}{\mathrm{e}}\right)^n 2^{a-3/2} \mathrm{e}^{1/z} \left\{1 - \frac{1+6(a-2)(a-1+2z^{-1})+6z^{-2}}{24n} + \mathrm{O}(n^{-2})\right\}, \tag{1.3}$$

holding uniformly on compact z-plane excluding the origin. In a review of Grosswald's book, Wimp [12] remarked that "What are lacking are uniform asymptotic estimates valid near $z=0$". Wimp's remark prompted us to reconsider the asymptotic behavior of $y_n(z;a)$ as $n \to \infty$. Another motivation for our study came from an open problem listed in the Appendix of [5], which asks the two-term expansion in (1.3) be developed into a complete asymptotic series.

A reason for the interest in the behavior of $y_n(z;a)$ for z near the origin is that zeros of these polynomials all lie in a neighborhood of $z=0$. For instance, it is known that for $a \geqslant 2$, all zeros $z_{n,k}(a)$ of $y_n(z;a)$ lie in the semi-annulus defined by the inequalities

$$\frac{2}{n(n+a-1)} \leqslant |z_{n,k}(a)| \leqslant \frac{2}{n+a-1}, \quad \mathrm{Re}\, z_{n,k}(a) < 0; \tag{1.4}$$

see [5, p. 82]. From (1.4), it is readily seen that the domain of validity of the two-term expansion in (1.3) does not contain the most interesting region where all the zeros are located. For more recent information concerning the zeros of these polynomials, we refer to [4, 2, 1].

It has been long recognized that the generalized Bessel polynomials are closely related to the Padé approximants to e^z; see, e.g., [5]. In the last of a series of papers on the zeros and poles of Padé approximants to e^z, Saff and Varga [9] investigated the asymptotic behavior of the integral

$$\begin{aligned} I_n(t) &= \int_0^\infty \mathrm{e}^{-u}(u+t)^n u^{v(n)} \mathrm{d}u \\ &= [n+v(n)]^{n+v(n)+1} \int_0^\infty \mathrm{e}^{[n+v(n)]h(u,\eta)} \,\mathrm{d}u \end{aligned} \tag{1.5}$$

as $n \to \infty$, where $v(n)$ is a function of $n, \eta = t/(n+v(n))$ and

$$h(u,\eta) = h(u) = -u + \frac{n}{n+v(n)}\log(u+\eta) + \frac{v(n)}{n+v(n)}\log u. \tag{1.6}$$

From the integral representation (1.1), it is evident that the result of Saff and Varga can be applied directly to the generalized Bessel polynomials. To describe their result, we assume that the limit

$$\lim_{n\to\infty} \frac{v(n)}{n} = \sigma, \quad 0 < \sigma < \infty, \tag{1.7}$$

exists, and define two complex numbers t_σ^+ and t_σ^- by

$$t_\sigma^\pm \equiv [(1-\sigma) \pm 2\sqrt{\sigma}i]/(1+\sigma) = \exp\left\{\pm \mathrm{i}\cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)\right\}. \tag{1.8}$$

Note that these numbers have modulus unity. Consider the complex plane $\mathbb{C}$ slit along the two rays

$$R_\sigma \equiv \{t\colon\ t = t_\sigma^+ + \mathrm{i}\tau \text{ or } t = t_\sigma^- - \mathrm{i}\tau,\ \tau \geqslant 0\}. \tag{1.9}$$

Saff and Varga proved that for $t \in \mathbb{C} \setminus (\{0\} \cup R_\sigma)$, the asymptotic behavior of $I_n(t)$ is given by one of the following three formulas:

$$I_n(t) = [n+v(n)]^{n+v(n)+1}\, I_n^+(\eta), \tag{1.10}$$

$$I_n(t) = [n+v(n)]^{n+v(n)+1}\, I_n^-(\eta), \tag{1.11}$$

$$I_n(t) = [n+v(n)]^{n+v(n)+1}\, \{I_n^+(\eta) + I_n^-(\eta)\}, \tag{1.12}$$

where

$$I_n^\pm(\eta) = \mathrm{e}^{-[n+v(n)]u^\pm(\eta)}[u^\pm(\eta)+\eta]^n \times \left|\frac{2\pi}{[n+v(n)]h''(u^\pm(\eta),\eta)}\right|^{1/2} \mathrm{e}^{\mathrm{i}\theta^\pm}\left\{1 + \mathrm{O}\left(\frac{1}{n+v(n)}\right)\right\}, \tag{1.13}$$

$$u^\pm(\eta) = \tfrac{1}{2}[1 - \eta \pm g(\eta)] \tag{1.14}$$

are the two saddle points of the phase function $h(u,\eta)$ in (1.6),

$$g(\eta) = \left\{1 + \eta^2 - 2\eta\frac{n-v(n)}{n+v(n)}\right\}^{1/2}, \tag{1.15}$$

$$\theta^\pm = \tfrac{1}{2}[\pi - \arg h''(u^\pm(\eta),\eta)]. \tag{1.16}$$

The O-symbol in (1.13) is uniform with respect to z in any compact subset of $\mathbb{C}\setminus(\{0\}\cup R_\sigma)$. Although this result of Saff and Varga is sufficient for their investigation of the location of zeros and poles of the Padé approximants to e^z, the information provided by (1.10)–(1.12) is still incomplete as far as the generalized Bessel polynomials are concerned. The reason is that for z in a given region of the complex plane $\mathbb{C}$, it does not specify exactly which of these three formulas holds. Furthermore, the region of validity, so far established, excludes the rays R_σ. If $z_n^\pm(a)$ denote the zeros of $y_n(z;a)$ with the largest real part in the upper and lower half of the z-plane, then, as we shall see later

(Section 4), the limits of $t_n^{\pm}(a)=2/z_n^{\pm}(a)$, as $n\to\infty$, belong to R_1 (In the case of the generalized Bessel polynomials, $\sigma=1$ in (1.7) and $t=2/z$ in (1.5).) Thus, regions close to the rays R_1 should be of great interest in the study of the asymptotic behavior of the zeros of $y_n(z;a)$.

In this paper, we shall first derive infinite asymptotic expansions for $y_n(z;a)$, which hold uniformly for $(n+1)z\equiv\alpha$ bounded away from $\alpha=\pm\mathrm{i}$. (Note that when α is finite, z tends to zero.) Then, we shall construct uniform asymptotic expansions for $y_n(z;a)$ in neighborhoods of $\alpha=\pm\mathrm{i}$. These expansions involve the Airy function and its derivative. Finally, we shall use these uniform approximations to study the asymptotic behavior of the zeros of $y_n(z;a)$ near $\alpha=\mathrm{i}$.

2. Expansions of $y_n(z;a)$ in domains not containing 0 and $\pm i/(n+1)$

From the generating function in (1.2), we have the integral representation

$$y_n(z;a)=\frac{n!}{2\pi\mathrm{i}}2^{a-2}\int_{C'}(1-2tz)^{-1/2}[1+(1-2tz)^{1/2}]^{2-a}\exp\left\{\frac{1-(1-2tz)^{1/2}}{z}\right\}\frac{\mathrm{d}t}{t^{n+1}},$$

where C' is a closed contour enclosing $t=0$, but not $t=1/2z$. Let $\zeta=(1-2tz)^{1/2}$ and $\alpha=(n+1)z$. Then, the above equation becomes

$$y_n(z;a)=-2^{a-1}n!(2z)^n\mathrm{e}^{1/z}\frac{1}{2\pi\mathrm{i}}\int_C g(\zeta)\mathrm{e}^{(n+1)f(\zeta,\alpha)}\mathrm{d}\zeta, \tag{2.1}$$

where C is the image of C', i.e., a closed curve in the ζ-plane enclosing $\zeta=1$ but excluding $\zeta=0$,

$$g(\zeta)=(1+\zeta)^{2-a} \tag{2.2}$$

and

$$f(\zeta,\alpha)=-\frac{\zeta}{\alpha}-\log(1-\zeta^2). \tag{2.3}$$

In the following, we shall first discuss in detail the phase function $f(\zeta,\alpha)$, and then use the classical method of steepest descent to find the asymptotic expansions of $y_n(z;a)$. Due to the fact that α is a complex parameter, the problem is considerably more complicated than it would be otherwise. Since $y_n(\bar{z};a)=y_n(z;a)$, we may without loss of generality restrict our attention to the upper half of the complex z-plane.

2.1. Properties of $f(\zeta,\alpha)$

The phase function $f(\zeta,\alpha)$ has only two saddle points

$$\zeta_{\pm}(\alpha)=-\alpha\pm\sqrt{\alpha^2+1}, \tag{2.4}$$

i.e., zeros of the derivative $\partial f/\partial\zeta$. We cut the complex α-plane along the rays

$$\Gamma^{\pm}=\{\alpha\colon\ \alpha=\pm\mathrm{i}\tau,\ \tau\geqslant 1\}. \tag{2.5}$$

On the edges of the cut Γ^+, we define

$$\sqrt{\alpha^2+1}=\pm\mathrm{i}\sqrt{\tau^2-1}, \tag{2.6}$$

where $\alpha = 0^{\pm} + \mathrm{i}\tau$ and $\tau \geqslant 1$. When $\alpha = \mathrm{i}$, the two saddle points $\zeta_{\pm}(\alpha)$ coincide, and we have $\zeta_{+}(\mathrm{i}) = \zeta_{-}(\mathrm{i}) = -\mathrm{i}$. The following result gives some idea about the locations of $\zeta_{\pm}(\alpha)$ in the ζ-plane.

Lemma 1. *For α in the upper half of the cut plane, we have*

$$-\tfrac{1}{2}\pi \leqslant \arg \zeta_{+}(\alpha) \leqslant 0, \qquad -\pi \leqslant \arg \zeta_{-}(\alpha) \leqslant -\tfrac{1}{2}\pi.$$

Proof. Let $\alpha = a + \mathrm{i}b$, $b > 0$. Then

$$\sqrt{\alpha^2+1} = (1 + a^2 - b^2 + \mathrm{i}2ab)^{1/2} = \left(\sqrt{(1+a^2-b^2)^2 + 4a^2b^2}\,\mathrm{e}^{\mathrm{i}\theta}\right)^{1/2},$$

where $\theta = \arg(1 + a^2 - b^2 + \mathrm{i}2ab)$. Clearly, if $1 + a^2 - b^2 = 0$ then $\theta = \pm\frac{1}{2}\pi$, depending on whether a is positive or negative. Otherwise,

$$\theta = \begin{cases} \arctan\dfrac{2ab}{1+a^2-b^2} & \text{if } \ 1+a^2-b^2 > 0, \\ \pi + \arctan\dfrac{2ab}{1+a^2-b^2} & \text{if } \ 1+a^2-b^2 < 0 \ \text{ and } \ a > 0, \\ -\pi + \arctan\dfrac{2ab}{1+a^2-b^2} & \text{if } \ 1+a^2-b^2 < 0 \ \text{ and } \ a < 0. \end{cases}$$

In the case when $1 + a^2 - b^2 > 0$, straightforward calculation shows

$$\begin{aligned} \mathrm{Re}\sqrt{\alpha^2+1} &= [(1+a^2-b^2)^2 + 4a^2b^2]^{1/4}\cos\tfrac{1}{2}\theta \\ &= \frac{1}{\sqrt{2}}\left\{\sqrt{(1+a^2-b^2)^2+4a^2b^2} + (1+a^2-b^2)\right\}^{1/2} \geqslant |a|, \end{aligned}$$

and

$$\begin{aligned} |\mathrm{Im}\sqrt{\alpha^2+1}| &= [(1+a^2-b^2)^2 + 4a^2b^2]^{1/4}|\sin\tfrac{1}{2}\theta| \\ &= \frac{1}{\sqrt{2}}\left\{\sqrt{(1+a^2-b^2)^2+4a^2b^2} - (1+a^2-b^2)\right\}^{1/2} \leqslant b. \end{aligned}$$

Hence, $\mathrm{Re}\,\zeta_{+}(\alpha) \geqslant 0$, $\mathrm{Im}\,\zeta_{+}(\alpha) \leqslant 0$, $\mathrm{Re}\,\zeta_{-}(\alpha) \leqslant 0$ and $\mathrm{Im}\,\zeta_{-}(\alpha) \leqslant 0$. The other cases are proved in a similar manner. □

The next result concerns some special curves in the complex α-plane, which will be used later in the discussion.

Lemma 2. *Each of the equations*

$$\mathrm{Re}\, f(\zeta_{+}, \alpha) = \mathrm{Re}\, f(\zeta_{-}, \alpha), \tag{2.7}$$

$$\mathrm{Im}\, f(\zeta_{+}, \alpha) = \mathrm{Im}\, f(\zeta_{-}, \alpha) \tag{2.8}$$

determines three solution curves in the neighborhood of $\alpha = \mathrm{i}$. The argument of the tangent lines to these curves at $\alpha = \mathrm{i}$ are $\pm\frac{1}{6}\pi$, $\pm\frac{1}{2}\pi$ and $\pm\frac{5}{6}\pi$. In the neighborhood of $\alpha = 0$, Eq. (2.8) gives two solution curves, both of which have horizontal tangent lines.

Proof. Let

$$\begin{aligned}\varphi(\alpha) &\equiv f(\zeta_+,\alpha) - f(\zeta_-,\alpha)\\ &= -2\left[\frac{\sqrt{\alpha^2+1}}{\alpha} + \log(\sqrt{\alpha^2+1} - \alpha) + \mathrm{i}\frac{\pi}{2}\right].\end{aligned} \tag{2.9}$$

In the neighborhood of $\alpha = \mathrm{i}$, we have

$$\begin{aligned}\varphi(\alpha) &= -\tfrac{4}{3}(1+\mathrm{i})(\alpha - \mathrm{i})^{3/2} + \mathrm{O}\{(\alpha - i)^2\}\\ &= -\tfrac{4}{3}\sqrt{2}r^{3/2}\mathrm{e}^{\mathrm{i}(\pi/4+3\theta/2)} + \mathrm{O}(r^2),\end{aligned} \tag{2.10}$$

where $\alpha = \mathrm{i} + r\mathrm{e}^{\mathrm{i}\theta}$. Eq. (2.7) is equivalent to Re $\varphi(\alpha) = 0$. Hence

$$\cos(\tfrac{1}{4}\pi + \tfrac{3}{2}\theta) + \mathrm{O}(r^{1/2}) = 0.$$

Letting $r \to 0$, one readily sees that the tangent lines to the curves defined by (2.7) have arguments given by $\theta = \frac{1}{6}\pi$, $-\frac{1}{2}\pi$ and $-\frac{7}{6}\pi$. Similarly, $\mathrm{Im}\,\varphi(\alpha) = 0$ yields

$$\sin(\tfrac{1}{4}\pi + \tfrac{3}{2}\theta) + \mathrm{O}(r^{1/2}) = 0.$$

Thus, the tangent lines to the curves defined by (2.8) have arguments given by $\theta = -\frac{1}{6}\pi$, $\frac{1}{2}\pi$ and $-\frac{5}{6}\pi$. In the neighborhood of $\alpha = 0$, we have

$$\varphi(\alpha) = -2\left[\frac{1}{\alpha} + \frac{\mathrm{i}}{2}\pi + \mathrm{O}(\alpha)\right].$$

Writing $\alpha = a + \mathrm{i}b$, equation (2.8) leads to $b = \frac{1}{2}\pi a^2 + \mathrm{o}(a^2)$. Clearly, for each value of b, there are two values of a, i.e., near the origin there are two solution curves, and the tangent lines to these curves are horizontal. □

Let C_1, C_2 and C_3 denote the solution curves determined by Eq. (2.7), and let Γ_1, Γ_2 and Γ_3 denote the solution curves determined by Eq. (2.8). These curves are depicted in Fig. 1 below. Note that the curves Γ_1, Γ_2 and Γ_3 divide the upper half of the α-plane into three regions R_1, R_2 and R_3.

2.2. Steepest descent paths of the phase function

In this subsection, we shall discuss the steepest descent paths of the phase function $f(\zeta,\alpha)$ in (2.3). As we shall see later, there are three types of such curves, corresponding to the three regions R_1, R_2 and R_3 in Fig. 1. By a steepest path of $f(\zeta,\alpha)$ through the saddle point ζ_+(or ζ_-), we shall mean a curve in the ζ-plane passing through ζ_+(or ζ_-) along which $\mathrm{Im}\, f(\zeta,\alpha)$ remains constant. If Re $f(\zeta,\alpha)$ decreases when ζ moves away from the saddle point along a steepest path in both directions,

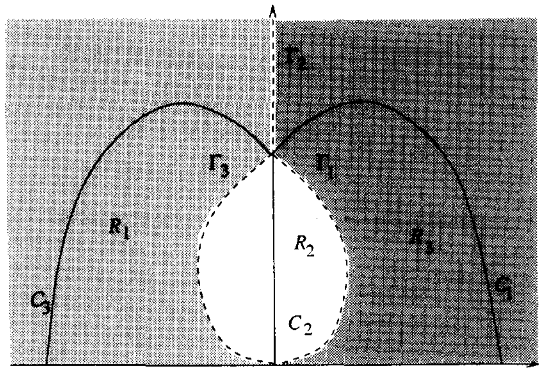

Fig. 1. Solution curves of (2.7) and (2.8).

then that path is called a steepest descent path. Otherwise, it is called a steepest ascent path. For a more detailed discussion of this topic, we refer to [13, pp. 84–90]. Put $\alpha = a + ib$ and $\zeta = x + iy$. Then (2.3) gives

$$\operatorname{Im} f(\zeta,\alpha) = \frac{ay - bx}{a^2 + b^2} - \arg(1 - \zeta^2).$$

Now, let ζ tends to infinity along a steepest path Γ. The last equation shows that the slope of Γ approaches b/a; i.e., the direction of Γ is approximately equal to that of the line segment from the origin to the point α.

First, we consider a few special values of α, namely, when α is real or purely imaginary. The steepest descent paths in these cases are shown in Fig. 2(a)–(f). The dotted lines in these figures are used to indicate that these curves are on a different Riemann surface.

Next, we consider the general case. Let $\alpha = \alpha_1$ be a point in the region R_1 depicted in Fig. 1, and let γ_1 be a curve connecting α_1 to -1 and completely contained in R_1. In Fig. 2(f), it is shown that when $\alpha = -1$, the contour $\Gamma_+(\alpha)$ embraces the contour $\Gamma_-(\alpha)$. Since $\operatorname{Im} f(\zeta_+,\alpha) \neq \operatorname{Im} f(\zeta_-,\alpha)$ as α moves from -1 to α_1, contour $\Gamma_+(\alpha_1)$ must also embrace contour $\Gamma_-(\alpha_1)$. Thus the two curves $\Gamma_+(\alpha)$ and $\Gamma_-(\alpha)$ are as depicted in Fig. 3.

Similarly, if α_2 is a point in the region R_2, then we let γ_2 be a curve in R_2 which connects α_2 to $i/2$. When $\alpha = i/2$, it is shown in Fig. 2(d) that contours Γ_+ and Γ_- are seperate from each other. Hence, by the same argument as above, $\Gamma_+(\alpha_2)$ and $\Gamma_-(\alpha_2)$ are seperate from each other; see Fig. 4. If $\alpha_3 \in R_3$, then we choose a curve γ_3 in R_3 connecting α_3 to 1. Since $\Gamma_-(1)$ embraces $\Gamma_+(1)$, by the same reasoning, we also have $\Gamma_-(\alpha_3)$ embracing $\Gamma_+(\alpha_3)$; see Fig. 5.

Finally, if α lies on the curves Γ_1 and Γ_3 in Fig. 1, then $\operatorname{Im} f(\zeta_+,\alpha) = \operatorname{Im} f(\zeta_-,\alpha)$, and hence the saddle points ζ_+ and ζ_- lie on each others steepest path; see Figs. 6 and 7.

To obtain the asymptotic expansions of the generalized Bessel polynomials, we need one more preliminary result concerning the slopes of the steepest descent paths at the saddle points.

Lemma 3. *The curves defined by the equation*

$$\operatorname{Im} \frac{1 + \zeta_+^2}{(1 - \zeta_+^2)^2} = 0 \tag{2.11}$$

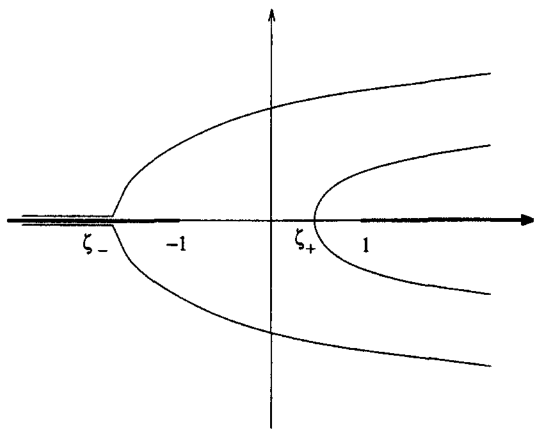

Fig. 2. (a) Steepest descent paths, when $\alpha = a > 0$ (here, $0 < \zeta_+ < 1$ and $\zeta_- < 1$).

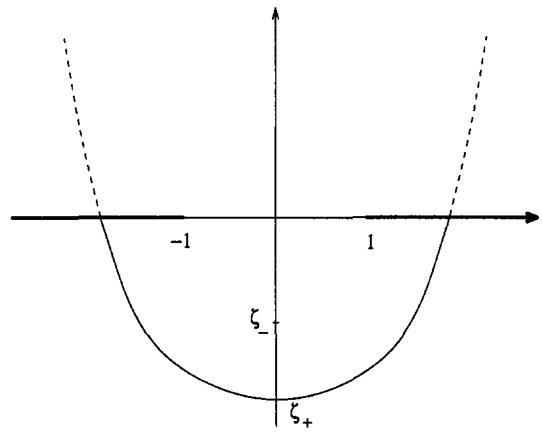

Fig 2. (b) Steepest descent path, when $\alpha = 0^- + ib$ and $b > 1$ (here, $\zeta_\pm = i(-b \mp \sqrt{b^2-1})$).

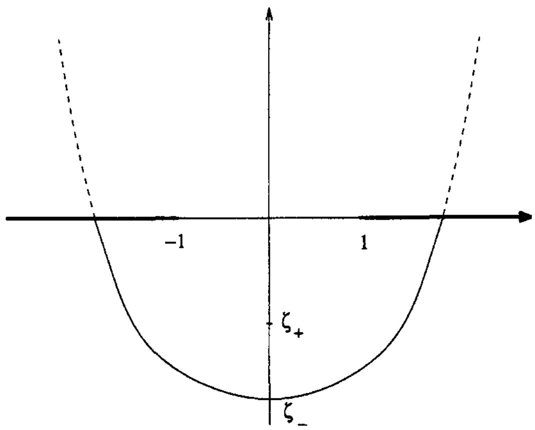

Fig. 2. (c) Steepest descent path, when $\alpha = 0^+ + ib$ and $b > 1$ (here, $\zeta_\pm = i(-b \pm \sqrt{b^2-1})$).

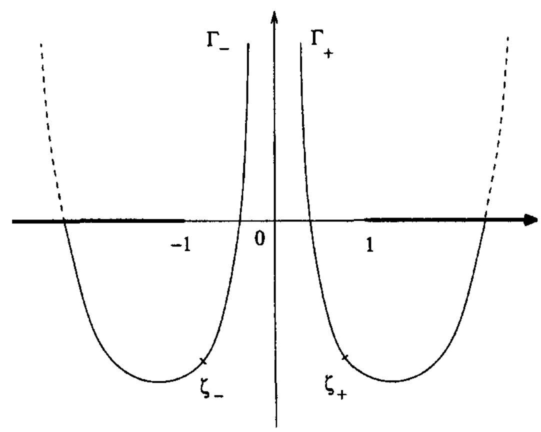

Fig. 2. (d) Steepest descent paths, when $\alpha = ib$ and $0 < b < 1$ (here, $\zeta_\pm = \pm\cos\theta - i\sin\theta$, $\theta = \sin^{-1} b$).

divide the upper half of the α-plane into two parts K_1 and K_2 as shown in Fig. 8. *Put*

$$\sigma_+(\alpha) = \operatorname{Re}\frac{1+\zeta_+^2}{(1-\zeta_+^2)^2} \quad \textit{and} \quad \tau_+(\alpha) = \operatorname{Im}\frac{1+\zeta_+^2}{(1-\zeta_+^2)^2}. \tag{2.12}$$

If $k_+(\alpha)$ denotes the slope of the steepest descent path at the saddle point ζ_+, then

$$k_+(\alpha) = \begin{cases} \dfrac{\sigma_+(\alpha))}{\tau_+(\alpha)} + \left\{\left[\dfrac{\sigma_+(\alpha)}{\tau_+(\alpha)}\right]^2 + 1\right\}^{1/2}, & \alpha \in K_1, \\[2ex] \dfrac{\sigma_+(\alpha)}{\tau_+(\alpha)} - \left\{\left[\dfrac{\sigma_+(\alpha)}{\tau_+(\alpha)}\right]^2 + 1\right\}^{1/2}, & \alpha \in K_2. \end{cases} \tag{2.13}$$

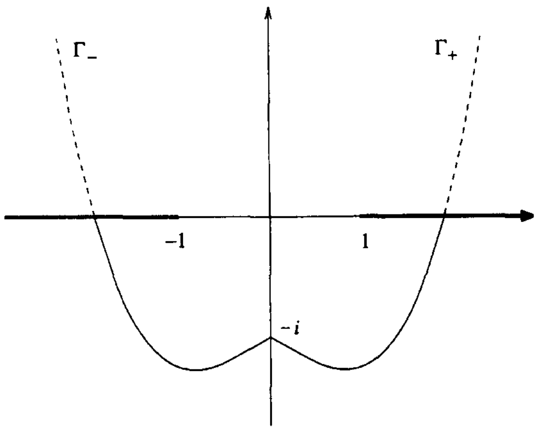

Fig. 2. (e) Steepest descent path, when $\alpha = \mathrm{i}$ (here, $\zeta_\pm = -\mathrm{i}$).

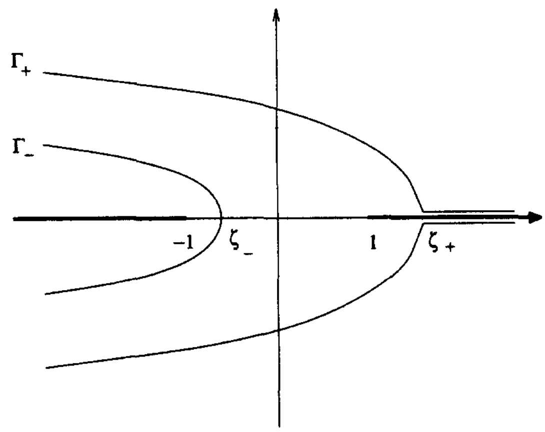

Fig. 2. (f) Steepest descent paths, when $\alpha = a < 0$ (here, $-1 < \zeta_- < 0$ and $\zeta_+ > 1$).

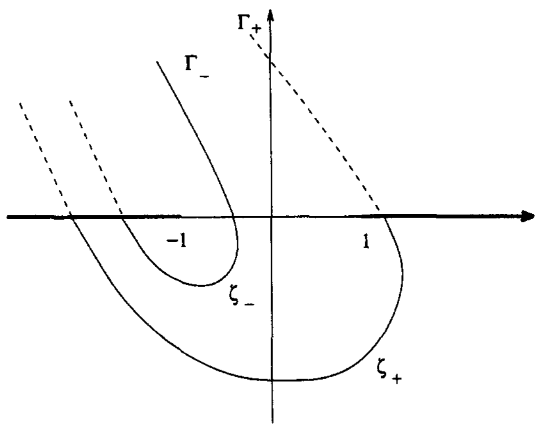

Fig. 3. Steepest descent paths Γ_+ and Γ_-, when $\alpha \in R_1$.

Similarly, the curves defined by the equation

$$\operatorname{Im}\frac{1+\zeta_-^2}{(1-\zeta_-^2)^2} = 0 \tag{2.14}$$

divide the upper half of the α-plane into two parts K_1' and K_2' as shown in Fig. 9. If $k_-(\alpha)$ denotes the slope of the steepest descent path at the saddle point ζ_-, then

$$k_-(\alpha) = \begin{cases} \dfrac{\sigma_-(\alpha)}{\tau_-(\alpha)} + \left\{\left[\dfrac{\sigma_-(\alpha)}{\tau_-(\alpha)}\right]^2 + 1\right\}^{1/2}, & \alpha \in K_1', \\[2ex] \dfrac{\sigma_-(\alpha)}{\tau_-(\alpha)} - \left\{\left[\dfrac{\sigma_-(\alpha)}{\tau_-(\alpha)}\right]^2 + 1\right\}^{1/2}, & \alpha \in K_2'. \end{cases} \tag{2.15}$$

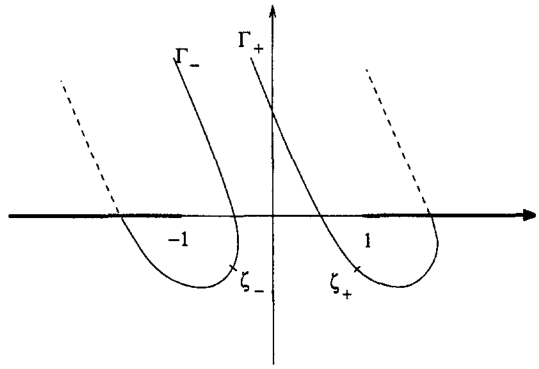

Fig. 4. Steepest descent paths Γ_+ and Γ_-, when $\alpha \in R_2$.

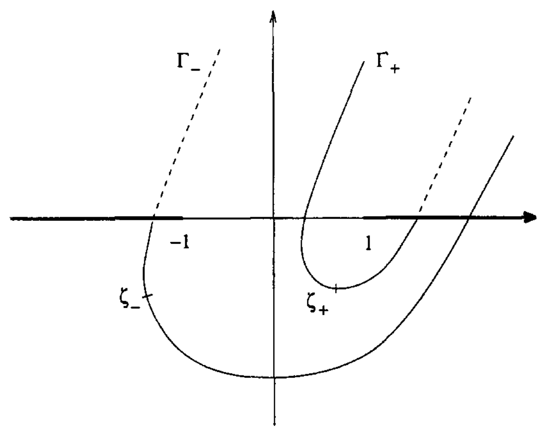

Fig. 5. Steepest descent paths Γ_+ and Γ_-, when $\alpha \in R_3$.

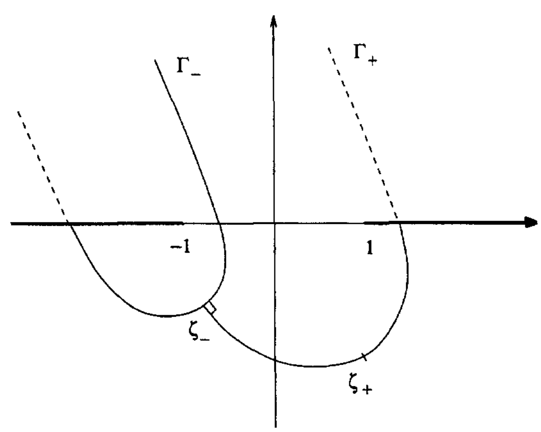

Fig. 6. Steepest descent paths, when $\alpha \in \Gamma_3$.

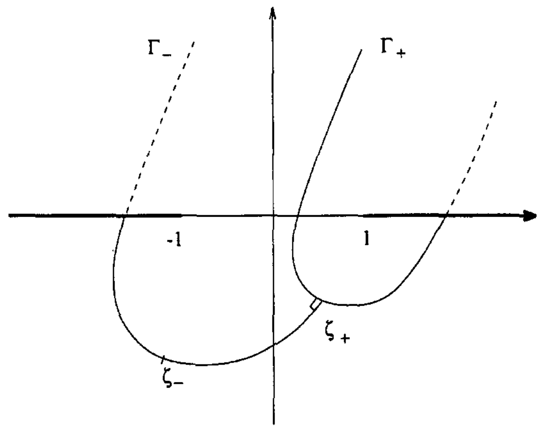

Fig. 7. Steepest descent paths, when $\alpha \in \Gamma_1$.

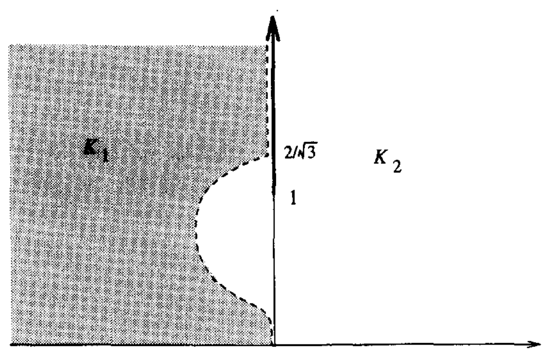

Fig. 8. Regions K_1 and K_2 in the α-plane.

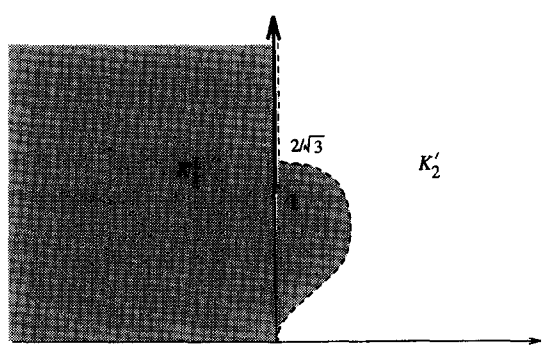

Fig. 9. Regions K_1' and K_2' in the α-plane.

where

$$\sigma_-(\alpha) = \mathrm{Re}\frac{1+\zeta_-^2}{(1-\zeta_-^2)^2} \quad \textit{and} \quad \tau_-(\alpha) = \mathrm{Im}\,\frac{1+\zeta_-^2}{(1-\zeta_-^2)^2}. \tag{2.16}$$

Proof. Let

$$\Psi_+(\alpha) = \frac{1+\zeta_+^2}{(1-\zeta_+^2)^2} = \frac{\sqrt{\alpha^2+1}}{2\alpha^2(\sqrt{\alpha^2+1}-\alpha)}. \tag{2.17}$$

Clearly, $\Psi_+(\alpha) = \sigma_+(\alpha) + i\tau_+(\alpha)$. When $\alpha \in \mathbb{R}$ or $\alpha = 0^{\pm} + ib$ with $b > 1$, it is readily seen from (2.6) that $\mathrm{Im}\,\Psi_+(\alpha) = 0$. Let $\alpha_0 = i2/\sqrt{3}$, and expand $\Psi_+(\alpha)$ into a Taylor series at α_0. (The reason for looking at the point $i2/\sqrt{3}$ will become apparent later in the proof.) For α on the left-hand side of the imaginary axis, the result is

$$\Psi_+(\alpha) = -0.125 - 0.84374991(\alpha-\alpha_0)^2 + \mathrm{O}\{(\alpha-\alpha_0)^3\}.$$

Hence, in the neighborhood of α_0 on the left-hand side of the imaginary axis, Eq. (2.14) has another solution curve, in addition to the one along the imaginary axis, whose slope at α_0 is zero. For α on the right-hand side of the imaginary axis, the Taylor expansion of $\Psi_+(\alpha)$ at α_0 is

$$\Psi_+(\alpha) = 0.375 - 1.299038103i(\alpha-\alpha_0) + \mathrm{O}\{(\alpha-\alpha_0)^2\},$$

from which it follows that the only solution curve of Eq. (2.14) in the neighborhood of α_0 on the right half plane is the imaginary axis. On the other hand, in the neighborhood of the origin, we have the Laurent expansion

$$\Psi(\alpha) = \frac{1}{2\alpha^2} + \mathrm{O}\left(\frac{1}{\alpha}\right).$$

Hence, in the neighborhood of $\alpha = 0$ in the upper half of the α-plane, Eq. (2.14) has two solution curves; one is the real axis, and the other has a vertical tangent at $\alpha = 0$. The curves defined by Eq. (2.14) are depicted in Fig. 8. These curves divide the upper half of the α-plane into two parts K_1 and K_2.

In the neighborhood of the saddle point ζ_+, the phase function $f(\zeta,\alpha)$ has the Taylor expansion

$$f(\zeta,\alpha) = f(\zeta_+,\alpha) + \frac{1+\zeta_+^2}{(1-\zeta_+^2)^2}(\zeta-\zeta_+)^2 + \mathrm{O}\{(\zeta-\zeta_+)^3\}. \tag{2.18}$$

Note that the coefficient of the second term on the right-hand side is exactly the function $\Psi_+(\alpha)$ defined in (2.17). From (2.18), it can be easily shown that the slopes of the steepest paths at ζ_+ are

$$k_+(\alpha) = \frac{\sigma_+(\alpha)}{\tau_+(\alpha)} \pm \left\{\left[\frac{\sigma_+(\alpha)}{\tau_+(\alpha)}\right]^2 + 1\right\}^{1/2}. \tag{2.19}$$

To decide which sign we should choose, we let $\alpha = -\varepsilon + ib$ $(b > 1)$ and expand $\Psi_+(\alpha)$ into a Taylor series at $\alpha_0 = 0^- + ib$. The result is

$$\Psi_+(\alpha) = -\frac{\sqrt{b^2-1}}{2b^2(\sqrt{b^2-1}+b)}\left[1 - \frac{b^2-2+b\sqrt{b^2-1}}{b(b^2-1)}i\varepsilon + \cdots\right].$$

(Here, ε is a small parameter.) Since $\Psi_+(\alpha)=\sigma_+(\alpha)+i\tau_+(\alpha)$, the last equation gives

$$\frac{\sigma_+(\alpha)}{\tau_+(\alpha)} \approx -\frac{b(b^2-1)}{b^2-2+b\sqrt{b^2-1}}\frac{1}{\varepsilon} \quad (b>1). \tag{2.20}$$

The denominator has only one real zero, and it occurs at $b=2/\sqrt{3}$. (Note that $-2/\sqrt{3}$ is not a zero.) From (2.20), it is easily seen that $\sigma_+(\alpha)/\tau_+(\alpha)$ is negative when $b>2/\sqrt{3}$, and that $\sigma_+(\alpha)/\tau_+(\alpha)$ is positive when $1<b<2/\sqrt{3}$. We recall that the slope of the steepest descent path at ζ_+ is zero, when α lies on the left edge of the cut (i.e., when $\alpha=0^-+ib$ with $b>1$); see Fig. 2(c). Hence, we must choose $+$ sign in (2.19) when $b>2/\sqrt{3}$, and $-$ sign in (2.19) when $1<b<2/\sqrt{3}$. Since the slope of the steepest descent path at the saddle point ζ_+ depends continuously on α in the upper half of the cut α-plane, we must continue to choose $+$ sign in (2.19) when $\alpha\in K_1$, and $-$ sign in (2.19) when $\alpha\in K_2$. This establishes (2.13). The result for $k_-(\alpha)$ in (2.15) is proved in a similar manner, thus completing the proof of Lemma 3. □

2.3. Asymptotic expansions of $y_n(z;a)$

Let α be a point in the upper half of the α-plane bounded away from i and 0, and let $\alpha=(n+1)z$. For α in different regions R_1, R_2 and R_3 depicted in Fig. 1, we shall show that $y_n(z;a)$ has different asymptotic expansions.

Case I: $\alpha\in R_1$. Returning to (2.1), we put

$$I_n(\alpha)\equiv\frac{1}{2\pi i}\int_C g(\zeta)\,e^{(n+1)f(\zeta,\alpha)}\,d\zeta, \tag{2.21}$$

where C is a closed curve in the ζ-plane enclosing $\zeta=1$ but not $\zeta=0$. By Cauchy's theorem,

$$I_n(\alpha)=\frac{1}{2\pi i}\left(\int_{\Gamma_+}-\int_{\Gamma_-}\right)g(\zeta)\,e^{(n+1)f(\zeta,\alpha)}\,d\zeta,$$

where the contours Γ_+ and Γ_- are shown in Fig. 3 and both oriented in the counterclockwise direction. (It should be pointed out that the cuts in Fig. 3 are for the phase function $f(\zeta,\alpha)$, and not for the function $e^{(n+1)f(\zeta,\alpha)}$ which is analytic and single-valued for all ζ.) Using the steepest descent method [13, p. 84], we obtain

$$\begin{aligned} I_n(\alpha)\sim{}& \frac{1}{\pi i}\,e^{(n+1)f(\zeta_+,\alpha)}\sum_{s=0}^{\infty}C_{2s}^{(+)}\,\Gamma\left(s+\frac{1}{2}\right)(n+1)^{-s-(1/2)} \\ &-\frac{1}{\pi i}\,e^{(n+1)f(\zeta_-,\alpha)}\sum_{s=0}^{\infty}C_{2s}^{(-)}\,\Gamma\left(s+\frac{1}{2}\right)(n+1)^{-s-(1/2)}, \end{aligned} \tag{2.22}$$

where the coefficients C_{2s}^{+} and C_{2s}^{-} can be found in [13, p. 90]. In particular, we have

$$C_0^{(+)}=\frac{g(\zeta_+)}{\sqrt{2}[-f''(\zeta_+,\alpha)]^{1/2}} \quad\text{and}\quad C_0^{(-)}=\frac{g(\zeta_-)}{\sqrt{2}[-f''(\zeta_-,\alpha)]^{1/2}}, \tag{2.23}$$

where the branches of the square roots are chosen to satisfy

$$\tfrac{1}{2}\pi < \arg f''(\zeta_\pm,\alpha) + 2\arctan k_\pm(\alpha) < \tfrac{3}{2}\pi,$$

$k_\pm(\alpha)$ being the slopes of the steepest descent path at $\zeta_\pm$; see [13, p. 90].

Case II: $\alpha \in R_2 \cup R_3$. From (2.21), we have by Cauchy's theorem

$$I_n(\alpha) = \frac{1}{2\pi i}\int_{\Gamma_+} g(\zeta)\,e^{(n+1)f(\zeta_+,\alpha)}\,d\zeta, \tag{2.24}$$

where the contour Γ_+ is shown either in Fig. 4 or in Fig. 5. The method of steepest descent then gives

$$I_n(\alpha) \sim \frac{1}{\pi i}\,e^{(n+1)f(\zeta_+,\alpha)}\sum_{s=0}^{\infty} C_{2s}^{(+)}\,\Gamma\left(s+\frac{1}{2}\right)(n+1)^{-s-(1/2)}, \tag{2.25}$$

where C_0^+ is as given in (2.23).

Expansion (2.25) also holds when α lies on the curves Γ_3 and Γ_1. This can be seen from Figs. 6 and 7, respectively. If α lies on Γ_2, then we can see from Figs. 2(b) and (c) that the contribution to the asymptotic expansion of $I_n(\alpha)$ comes from either ζ_- or ζ_+, depending on whether α is on the left or right edge of the cut. In both cases, the path of integration begins at $+i\infty$, travels along the positive imaginary axis and the right half of the parabola, and passes through both saddle points ζ_+ and ζ_-. (Simple calculation also shows that $f(\zeta_+,\alpha) < f(\zeta_-,\alpha)$ when α lies on the left edge of the cut and $f(\zeta_+,\alpha) > f(\zeta_-,\alpha)$ when α lies on the right edge of the cut.) Since $\zeta_+(0^- + ib) = \zeta_-(0^+ + ib) = i(-b-\sqrt{b^2-1})$ for $b>1$, expansion (2.25) holds for α lying on both sides of the cut. (This is consistant with the fact that the function $I_n(\alpha)$ is analytic and single-valued in the upper half of the α-plane.) We summarize the above results in the following theorem.

Theorem A. *Let* $\alpha=(n+1)z$ *and* $\Pi^+=\{\alpha\in\mathbb{C}\colon \operatorname{Im}\alpha\geqslant 0, \alpha\neq 0$, *and* $\alpha\neq i\}$. *For* $\alpha\in\pi^+$, *the generalized Bessel polynomials have the asymptotic expansions*

$$\begin{aligned}
&-\frac{2^{1-a}}{n!}(2z)^{-n}e^{-1/z}\,y_n(z;a)\\
&\quad\sim \frac{1}{\pi i}\,e^{(n+1)f(\zeta_+,\alpha)}\sum_{s=0}^{\infty} C_{2s}^{(+)}\,\Gamma\left(s+\frac{1}{2}\right)(n+1)^{-s-(1/2)}\\
&\qquad -\frac{1}{\pi i}\,e^{(n+1)f(\zeta_-,\alpha)}\sum_{s=0}^{\infty} C_{2s}^{(-)}\,\Gamma\left(s+\frac{1}{2}\right)(n+1)^{-s-(1/2)},
\end{aligned} \tag{2.26}$$

if $\alpha\in R_1$, *and*

$$\begin{aligned}
&-\frac{2^{1-a}}{n!}(2z)^{-n}e^{-1/z}\,y_n(z;a)\\
&\quad\sim \frac{1}{\pi i}\,e^{(n+1)f(\zeta_+,\alpha)}\sum_{s=0}^{\infty} C_{2s}^{(+)}\,\Gamma\left(s+\frac{1}{2}\right)(n+1)^{-s-(1/2)}
\end{aligned} \tag{2.27}$$

if $\alpha\in R_2\cup R_3$, *or* $\alpha\in\Gamma_i$, $i=1,2,3$.

If α lies on the negative real-axis then expansion (2.26) takes a slightly different form, in view of the cut from $\zeta=1$ to $\zeta=\infty$ along the positive real axis (see Fig. 2(f)). By deforming the closed contour C in (2.21) into the steepest descent path in Fig. 2(f), we have by the steepest-descent method [13, pp. 84–94]

$$\begin{aligned} I_n(\alpha) \sim \frac{1}{\pi \mathrm{i}} \Bigg\{ & \mathrm{e}^{(n+1)f_-(\zeta_+,\alpha)} \sum_{k=0}^{\infty} C_{-,k}^{(+)} \Gamma\left(\frac{k+1}{2}\right)(n+1)^{-(k+1)/2} \\ & + \mathrm{e}^{(n+1)f_+(\zeta_+,\alpha)} \sum_{k=0}^{\infty} C_{+,k}^{(+)} \Gamma\left(\frac{k+1}{2}\right)(n+1)^{-(k+1)/2} \\ & - \mathrm{e}^{(n+1)f(\zeta_-,\alpha)} \sum_{s=0}^{\infty} C_{2s}^{(-)} \Gamma\left(s+\frac{1}{2}\right)(n+1)^{-s-(1/2)} \Bigg\}, \end{aligned} \tag{2.28}$$

where $f_+(\zeta,\alpha)$ and $f_-(\zeta,\alpha)$ denote, respectively, the values of $f(\zeta,\alpha)$ on the upper and lower edge of the cut, and where

$$C_{-,0}^{(+)} = \frac{g(\zeta_+)}{2^{3/2}[-f_-''(\zeta_+,\alpha)]^{1/2}}, \qquad C_{+,0}^{(+)} = \frac{g(\zeta_+)}{2^{3/2}[-f_+''(\zeta_+,\alpha)]^{1/2}}, \tag{2.29}$$

and $C_0^{(-)}$ is given as in (2.23). Note that other than $f(\zeta,\alpha)$ itself, all its derivatives are single-valued. Hence, in particular, $f_+''(\zeta_+,\alpha) = f_-''(\zeta_+,\alpha)$, i.e., $C_{-,0}^{(+)} = C_{+,0}^{(-)}$. On the upper and lower edges of the cut, we choose $\arg(\zeta-1)$ to be equal to 0 and 2π, respectively. With $-1=\mathrm{e}^{-\pi \mathrm{i}}$, we thus obtain

$$f_+(\zeta_+,\alpha) = \frac{\alpha-\sqrt{\alpha^2+1}}{\alpha} - \log(2\alpha^2 - 2\alpha\sqrt{\alpha^2+1}) + \mathrm{i}\pi, \tag{2.30}$$

and

$$f_-(\zeta_+,\alpha) = \frac{\alpha-\sqrt{\alpha^2+1}}{\alpha} - \log(2\alpha^2 - 2\alpha\sqrt{\alpha^2+1}) - \mathrm{i}\pi, \tag{2.31}$$

The logarithmic term in (2.30) and (2.31) is real.

An entirely analogous result can be derived for $I_n(\alpha)$, in the case when α lies on the positive real-axis.

When $z=\alpha/(n+1)$ is finite and bounded away from zero, it can be verified that the results (2.26) and (2.27) reduce to that of Doceř given in (1.3).

3. Uniform asymptotic expansions

When α approaches i, the two saddle points ζ_+ and ζ_- coalesce at $\zeta=-\mathrm{i}$, and $f''(\zeta_+,\alpha) = f''(\zeta_-,\alpha)=0$. As a result, the coefficients C_0^+ and C_0^- in (2.23) become infinite, and the expansions (2.22) and (2.25) fail to hold. In this section, we shall show that by using the method of Chester, Friedman and Ursell, we can construct an asymptotic expansion for $I_n(\alpha)$ which holds uniformly for α in a neighborhood of i. For a discussion of this method, see, e.g., [8, pp. 351–357] or [13, pp. 366–372].

In a neighborhood of $\zeta = -\mathrm{i}$, the phase function $f(\zeta,\alpha)$ has the Taylor expansion

$$f(\zeta,\alpha) = \frac{\mathrm{i}}{\alpha} - \log 2 - \left(\frac{1}{2}+\mathrm{i}\right)(\zeta+\mathrm{i}) - \frac{\mathrm{i}}{6}(\zeta+\mathrm{i})^3 + \cdots.$$

This suggests that we make the cubic change of variable

$$f(\zeta,\alpha) = \tfrac{1}{3}u^3 - A^2(\alpha)u + B(\alpha). \tag{3.1}$$

To make the mapping $\zeta \leftrightarrow u$ one-to-one and analytic, saddle points $\zeta_\pm$ of $f(\zeta,\alpha)$ must correspond to the saddle points $u_\pm = \pm A(t)$ of the cubic polynomial on the right-hand side of (3.1). Thus we set

$$-\frac{\zeta_+}{\alpha} - \log(1-\zeta_+^2) = \frac{u_+^3}{3} - A^2(\alpha)u_+ + B(\alpha),$$

$$-\frac{\zeta_-}{\alpha} - \log(1-\zeta_-^2) = \frac{u_-^3}{3} - A^2(\alpha)u_- + B(\alpha).$$

Solving these equations for $A(\alpha)$ and $B(\alpha)$, we obtain

$$A^3(\alpha) = \frac{3}{2}\left[\frac{\sqrt{\alpha^2+1}}{\alpha} + \log(-\mathrm{i}\alpha + \mathrm{i}\sqrt{\alpha^2+1})\right], \tag{3.2}$$

$$B(\alpha) = 1 - \log(2\alpha\mathrm{i}). \tag{3.3}$$

A comparison of (3.2) with (2.9) gives

$$A^3(\alpha) = -\tfrac{3}{4}[f(\zeta_+,\alpha) - f(\zeta_-,\alpha)] = -\tfrac{3}{4}\varphi(\alpha). \tag{3.4}$$

The mapping

$$\alpha \to \Omega = A^3(\alpha) \tag{3.5}$$

takes the three curves Γ_1, Γ_2 and Γ_3 in Fig. 1 to the real axis in the Ω-plane. With the aid of (2.10), it is also readily seen that the transformation (3.5) maps the regions R_1, R_2 and R_3 in Fig. 1 into three half planes; see Fig. 10 below, where the point $\Omega = 0$ corresponds to $\alpha = \mathrm{i}$, and

$$R_1 = \{\Omega\colon -2\pi < \arg\Omega < -\pi\}, \qquad R_2 = \{\Omega\colon -\pi < \arg\Omega < 0\}, \qquad R_3 = \{\Omega\colon 0 < \arg\Omega < \pi\}.$$

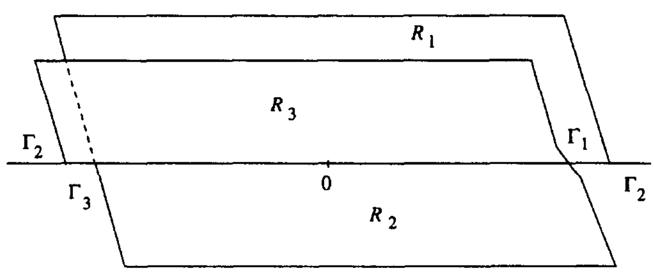

Fig. 10. Ω-plane.

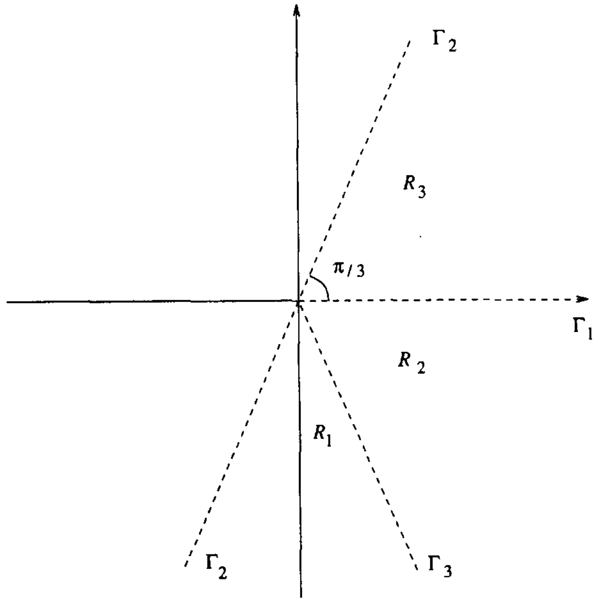

Fig. 11. Regions R_1, R_2 and R_3 in the $A(\alpha)$-plane.

Taking the cubic root

$$A(\alpha) = \Omega^{1/3} = \left\{ \frac{3}{2} \left[\frac{\sqrt{\alpha^2+1}}{\alpha} + \log(-\mathrm{i}\alpha + \mathrm{i}\sqrt{\alpha^2+1}) \right] \right\}^{1/3}, \tag{3.6}$$

the regions R_1, R_2, and R_3 become as shown in Fig. 11.

The functions $A(\alpha)$ and $B(\alpha)$ in (3.6) and (3.3) are univant in the upper half of the α-plane cut along the ray Γ^+ in (2.5). By a theorem of Chester, Friedman and Ursell [3], the mapping (3.1) has exactly one branch $u = u(\zeta, \alpha)$ which is uniformly regular for small $|\zeta + \mathrm{i}|$ and $|\alpha - \mathrm{i}|$, and on this branch, the points $\zeta = \zeta_\pm$ correspond to $u = \pm A(\alpha)$, respectively. Furthermore, for α near i, the correspondence $\zeta \leftrightarrow u$ is one-to-one. Hence, we recall that under a one-to-one analytic transformation, steepest descent paths are mapped into steepest descent paths; cf. [11]. We first investigate the steepest descent paths passing through the two saddle points of the cubic function $F(u,A) = \frac{1}{3}u^3 - A^2(\alpha)u + B(\alpha)$. Clearly, there are three directions in which the exponential function $\exp\{(n+1)F(u,A)\}$ decays most rapidly, namely, $\arg u = \frac{1}{3}\pi, -\frac{1}{3}\pi$ and π. A steepest descent path Γ passing through a saddle point is said to be of type A_1, if it tends to infinity in the two directions $\arg u = -\pi$ and $\arg u = -\frac{1}{3}\pi$; it is said to be of type A_2, if the directions of Γ tend to $\pm\frac{1}{3}\pi$. Similarly, we said that Γ is of type A_3, if it tends to infinity in the two directions $\arg u = \frac{1}{3}\pi$ and $\arg u = \pi$. There are also three directions in which $\exp\{(n+1)F(u,A)\}$ grows most rapidly; one is along the positive real axis, and the other $\arg u = \pm\frac{2}{3}\pi$. A steepest ascent path passing through a saddle point is said to be of type B_1, if it tends to infinity in the two directions $\arg u = \pm\frac{2}{3}\pi$; it is said to be of type B_2, if it tends to infinity along the rays $\arg u = 0$ and $\arg u = \frac{2}{3}\pi$; it is of type B_3, if it ends at infinity in the directions $\arg u = 0$ and $\arg u = -\frac{2}{3}\pi$.

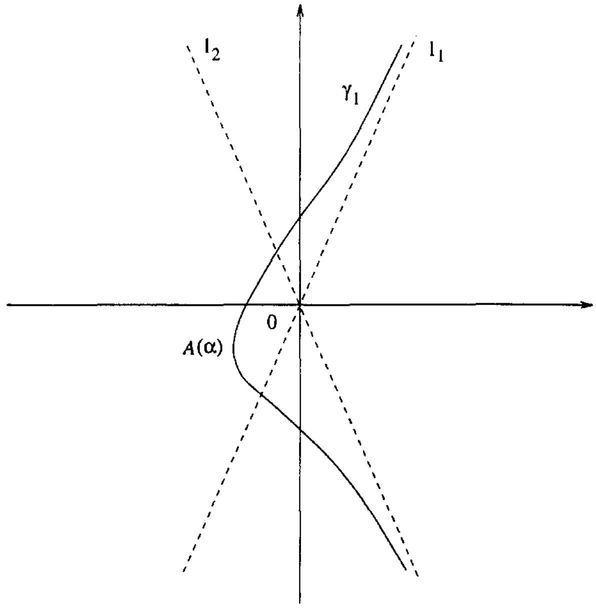

Fig. 12. The u-plane.

Lemma 4. *Divide the complex A-plane into three regions*

$$S_1 = \{A \in \mathbb{C}:\quad -\pi < \arg A < -\tfrac{1}{3}\pi\},$$
$$S_2 = \{A \in \mathbb{C}:\quad -\tfrac{1}{3}\pi < \arg A < \tfrac{1}{3}\pi\},$$
$$S_3 = \{A \in \mathbb{C}:\quad \tfrac{1}{3}\pi < \arg A < \pi\}.$$

A steepest descent path of the cubic function

$$F(u,A) = \frac{1}{3}u^3 - A^2(\alpha)u + B$$

passing through the saddle point $u = A(\alpha)$ *is of type* A_i, $i = 1,2,3$, *if* $A(\alpha) \in S_i$.

Proof. We shall consider only the case $i=1$; the other two cases can be proved in a similar manner. Let l_1 and l_2 be the two infinite lines $u = \tau e^{\pm i\pi/3}, \tau \in \mathbb{R}$. Set $u = s + it$ and $A(\alpha) = \sigma + i\tau$. The steepest (descent and ascent) pathes of $F(u,A)$ which pass through the saddle point $u = A(\alpha)$ are given by

$$-\tfrac{1}{3}t^3 - (\sigma^2 - \tau^2 - s^2)t - 2\sigma\tau s = -\tfrac{2}{3}(3\sigma^2\tau - \tau^3). \tag{3.7}$$

From (3.7), one can readily verify that the steepest paths cross each of the lines l_1, l_2 and the real axis at most once. (For instance, if we substitute $t = \sqrt{3}s$ in (3.7), we get a linear equation in s, which clearly has only one solution.) Let $A(\alpha) \in S_1$, and let γ_1 be the steepest descent path through $u = A(\alpha)$. If γ_1 is of type A_2 (see Fig. 12), then γ_1 crosses each of l_1 and the real axis once. Let γ_2 be the steepest ascent path passing through $u = A(\alpha)$. γ_2 may be of type B_1, B_2 or B_3. If γ_2 is of type B_1, then it crosses the real axis once. This contradicts the fact that γ_1 and γ_2 together can cross

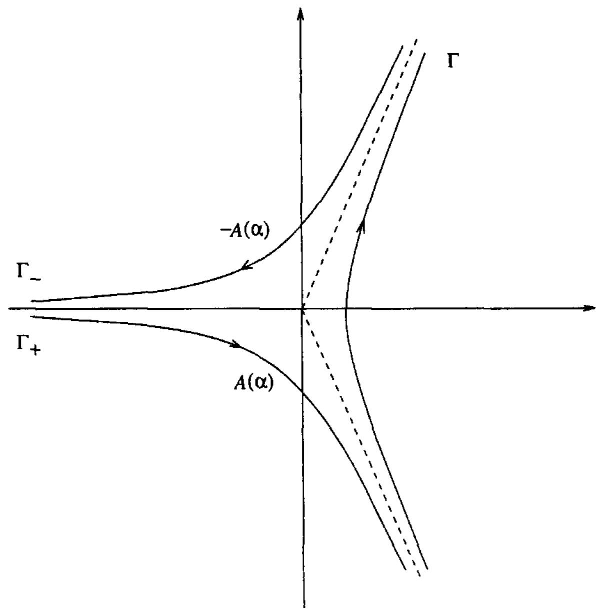

Fig. 13. Steepest descent paths in the u-plane, $\alpha \in R_1$.

the real axis at most once. If γ_2 is of type B_2 or B_3, then it must cross the line l_1 once, which is also not possible. So γ_1 is not of type A_2. By the same argument, one can prove that γ_1 is not of type A_3. Therefore, γ_1 must be of type A_1. □

Let D be a neighborhood of $\alpha = \mathrm{i}$. We first consider the case $\alpha \in D \cap R_1$. Under the mapping (3.6), α is mapped to a point $A(\alpha) \in R_1 \subset S_1$ (see Fig. 11 and Lemma 4) and $-A(\alpha) \in S_3$. Hence by Lemma 4, the steepest descent paths Γ_+ and Γ_- in Fig. 3 are mapped into the steepest descent paths of type A_1 and A_3, respectively; see Fig. 13 below. By Cauchy's theorem, we have from (3.1) and the equation following (2.21)

$$I_n(\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_+ - \Gamma_-} h_0(u) \exp\left\{ (n+1) \left[\frac{u^3}{3} - A^2(\alpha) u + B(\alpha) \right] \right\} \mathrm{d}u$$
$$= -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} h_0(u) \exp\left\{ (n+1) \left[\frac{u^3}{3} - A^2(\alpha) u + B(\alpha) \right] \right\} \mathrm{d}u, \tag{3.8}$$

where the directions of the paths $\Gamma_\pm$ and Γ are indicated by arrows in Fig. 13 and

$$h_0(u) = g(\zeta) \frac{\mathrm{d}\zeta}{\mathrm{d}u} = (1+\zeta)^{2-a} \alpha (1-\zeta^2) \frac{(u-A)(u+A)}{(\zeta - \zeta_+)(\zeta - \zeta_-)}. \tag{3.9}$$

From here on, the argument is quite standard, and can be found in, e.g., [13, pp. 378–382]. First we write

$$h_0(u) = \beta_0 + \gamma_0 u + [u^2 - A^2(\alpha)] \varphi_0(u), \tag{3.10}$$

from which it follows that

$$\beta_0 = \frac{1}{2}[h_0(A(\alpha)) + h_0(-A(\alpha))], \qquad \gamma_0 = \frac{1}{2A(\alpha)}[h_0(A(\alpha)) - h_0(-A(\alpha))] \tag{3.11}$$

and

$$\varphi_0(u) = \frac{h_0(u) - \beta_0 - \gamma_0 u}{u^2 - A^2(\alpha)}. \tag{3.12}$$

By inserting (3.11) into (3.12), we may rewrite $\varphi_0(u)$ in the form

$$\varphi_0(u) = \frac{h_0(u) - h_0(\pm A)}{u^2 - A^2} + \frac{h_0(\pm A) - h_0(\mp A)}{2(u^2 - A^2)}\left(1 \mp \frac{u}{A}\right). \tag{3.13}$$

From (3.13), it is clear that $\varphi_0(u)$ has removable singularities at $u = \pm A$ and consequently that it is analytic in the domain of $h_0(u)$. Substituting (3.10) in (3.8) and integrating term by term give

$$\begin{aligned} I_n(\alpha) = -\mathrm{e}^{(n+1)B(\alpha)} \Bigg[& \frac{\beta_0}{(n+1)^{1/3}} Ai\{(n+1)^{2/3}A^2(\alpha)\} \\ & - \frac{\gamma_0}{(n+1)^{2/3}} Ai'\{(n+1)^{2/3}A^2(\alpha)\} \\ & + \frac{1}{2\pi \mathrm{i}} \int_\Gamma (u^2 - A^2)\varphi_0(u) \exp\left\{(n+1)\left(\frac{u^3}{3} - A^2 u\right)\right\} \mathrm{d}u \Bigg], \end{aligned} \tag{3.14}$$

where $Ai(\cdot)$ denotes the Airy integral

$$Ai(x) = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}} \exp\left(\frac{u^3}{3} - xu\right) \mathrm{d}u, \tag{3.15}$$

$\mathscr{L}$ being any contour which begins at $\infty\,\mathrm{e}^{-\mathrm{i}\pi/3}$ and ends at $\infty\,\mathrm{e}^{\mathrm{i}\pi/3}$. To the integral in (3.14), we apply an integration by parts. This yields

$$\begin{aligned} I_n(\alpha) = -\mathrm{e}^{(n+1)B(\alpha)} \Bigg[& \frac{\beta_0}{(n+1)^{1/3}} Ai\{(n+1)^{2/3}A^2(\alpha)\} - \frac{\gamma_0}{(n+1)^{2/3}} Ai'\{(n+1)^{2/3}A^2(\alpha)\} \\ & - \frac{1}{2\pi \mathrm{i}(n+1)} \int_\Gamma \varphi_0'(u) \exp\left\{(n+1)\left(\frac{u^3}{3} - A^2(\alpha)u\right)\right\} \mathrm{d}u \Bigg]. \end{aligned} \tag{3.16}$$

Since $\varphi_0'(u)$ is analytic in u, the last integral is exactly of the same form as the original integral in (3.8). Hence the above procedure can be repeated. Define inductively, for $m = 0, 1, 2, \ldots,$

$$h_m(u) = \beta_m + \gamma_m u + (u^2 - A^2)\,\varphi_m(u), \qquad h_{m+1}(u) = \varphi_m'(u). \tag{3.17}$$

Then we obtain

$$\begin{aligned} I_n(\alpha) = -\mathrm{e}^{(n+1)B(\alpha)} \Bigg[& Ai\{(n+1)^{2/3}A^2(\alpha)\} \sum_{s=0}^{p-1} \frac{(-1)^s \beta_s}{(n+1)^{s+1/3}} \\ & - Ai'\{(n+1)^{2/3}A^2(\alpha)\} \sum_{s=0}^{p-1} \frac{(-1)^s \gamma_s}{(n+1)^{s+2/3}} + \varepsilon_p(n, \alpha) \Bigg], \end{aligned} \tag{3.18}$$

where

$$\varepsilon_p(n,\alpha)=\frac{(-1)^p}{(n+1)^p}\frac{1}{2\pi i}\int_\Gamma h_p(u)\exp\left\{(n+1)\left(\frac{u^3}{3}-A^2(\alpha)u\right)\right\}du. \tag{3.19}$$

From (3.17), it is easily seen that

$$\beta_m=\frac{1}{2}[h_m(A(\alpha))+h_m(-A(\alpha))], \qquad \gamma_m=\frac{1}{2A(\alpha)}[h_m(A(\alpha))-h_m(-A(\alpha))]. \tag{3.20}$$

The remainder $\varepsilon_p(n,\alpha)$ can be estimated in a manner similar to that given in [3], and the result is

$$|\varepsilon_p(n,\alpha)|\leqslant\frac{M_p}{(n+1)^{p+1/3}}|Ai\{(n+1)^{2/3}A^2(\alpha)\}|+\frac{N_p}{(n+1)^{p+2/3}}|Ai'\{(n+1)^{2/3}A^2(\alpha)\}|, \tag{3.21}$$

where M_p and N_p are constants, independent of n and α. Estimate (3.21) holds uniformly with respect to $\alpha\in D\cap R_1$, as long as α stays away from the boundary curves Γ_2 and Γ_3 in Fig. 1. Thus, in particular, α may tend to i along any curve properly contained in R_1. On account of (3.11) and (3.9), the two leading coefficients are given by

$$\begin{aligned}\beta_0=\frac{1}{2}\Bigg\{&\alpha(1+\zeta_+)^{2-a}(1-\zeta_+^2)\frac{A(\alpha)}{\sqrt{\alpha^2+1}}\left[\frac{1+\zeta_+^2}{(1-\zeta_+^2)^2}\frac{1}{A(\alpha)}\right]^{1/2}\\&+\alpha(1+\zeta_-)^{2-a}(1-\zeta_-^2)\frac{A(\alpha)}{\sqrt{\alpha^2+1}}\left[-\frac{1+\zeta_+^2}{(1-\zeta_+^2)^2}\frac{1}{A(\alpha)}\right]^{1/2}\Bigg\},\end{aligned} \tag{3.22}$$

$$\begin{aligned}\gamma_0=\frac{1}{2A(\alpha)}\Bigg\{&\alpha(1+\zeta_+)^{2-a}(1-\zeta_+^2)\frac{A(\alpha)}{\sqrt{\alpha^2+1}}\left[\frac{1+\zeta_+^2}{(1-\zeta_+^2)^2}\frac{1}{A(\alpha)}\right]^{1/2}\\&-\alpha(1+\zeta_-)^{2-a}(1-\zeta_-^2)\frac{A(\alpha)}{\sqrt{\alpha^2+1}}\left[\frac{1+\zeta_-^2}{(1-\zeta_-^2)^2}\frac{1}{A(\alpha)}\right]^{1/2}\Bigg\},\end{aligned} \tag{3.23}$$

where the branches of the square roots are chosen so that

$$\arg\left\{\pm\frac{1+\zeta_\pm^2}{(1-\zeta_\pm^2)^2}\frac{1}{A(\alpha)}\right\}=2\left\{\arctan k_\pm(\alpha)-\arctan\left[\frac{\operatorname{Re}A(\alpha)}{\operatorname{Im}A(\alpha)}-\left(1+\left[\frac{\operatorname{Re}A(\alpha)}{\operatorname{Im}A(\alpha)}\right]\right)^{1/2}\right]\right\}. \tag{3.24}$$

Next, we consider the case $\alpha\in D\cap R_i$, $i=2$ and 3. Under the transformation (3.6), α is mapped into a point $A(\alpha)\in R_i\subset S_2$; see Fig. 11. Hence, by Lemma 4, the steepest descent path Γ_+ in both Figs. 4 and 5 is mapped into the steepest descent path of type A_2 in the u-plane. That is, we again obtain the integral given in (3.8). As a result, (3.18) and (3.21) also hold in this case. To summarize, we have the following result.

Theorem B. *Let $\alpha=(n+1)z$ and D be a neighborhood of $\alpha=\mathrm{i}$. Then, for $\alpha\in D$, the generalized Bessel polynomials have the uniform asymptotic expansion*

$$y_n(z;a)=-2^{a-1}n!(2z)^n\mathrm{e}^{1/z+(n+1)B(\alpha)}\left[Ai\{(n+1)^{2/3}A^2(\alpha)\}\sum_{s=0}^{p-1}\frac{(-1)^s\beta_s}{(n+1)^{s+1/3}}\right.$$
$$\left.-Ai'\{(n+1)^{2/3}A^2(\alpha)\}\sum_{s=0}^{p-1}\frac{(-1)^s\gamma_s}{(n+1)^{s+2/3}}+\varepsilon_p(n,\alpha)\right], \tag{3.25}$$

where the remainder $\varepsilon_p(n,\alpha)$ satisfies the estimate (3.21).

When α is bounded away from i, $(n+1)^{1/3}A(\alpha)$ becomes unbounded as $n\to\infty$. By using the asymptotic expansions of the Airy function and its derivative (see [8, p. 392]), it can be shown that (3.25) reduces to (2.26) or (2.27), depending on whether $\alpha\in R_1$ or $\alpha\in R_2\cup R_3$.

4. Asymptotics of the zeros

We first recall the following two general properties of the zeros of the generalized Bessel polynomials. For a proof of these results, we refer to [5, p. 80; 4, Theorem 4.5].

Lemma 5. (a) *All zeros of $y_n(z;a)$ are simple.* (b) *For each real number a, there exists an integer $n_0=n_0(a)$ such that all zeros of $y_n(z;a)$ are in the open left half-plane for $n\geqslant n_0$.*

Since the coefficients of $y_n(z;a)$ are all real, the zeros of the generalized Bessel polynomials are located symmetrically with respect to the real axis. Hence we may, without loss of generality, consider only those zeros lying in the upper half plane. Let z_n^* be such a zero, and let $\alpha_n^*=(n+1)z_n^*$. From Theorem B, it follows that

$$\beta_0 Ai\{(n+1)^{2/3}A^2(\alpha_n^*)\}-\frac{\gamma_0}{(n+1)^{1/3}}Ai'\{(n+1)^{2/3}A^2(\alpha_n^*)\}+\varepsilon_1(n,\alpha)=0. \tag{4.1}$$

If $(n+1)^{2/3}A^2(\alpha_n^*)$ is bounded, then the second term in (4.1) is of smaller order of magnitude than the first. Hence, in view of (3.21), we have

$$Ai\{(n+1)^{2/3}A^2(\alpha_n^*)\}=\mathrm{O}(n^{-1/3}). \tag{4.2}$$

which in turn implies that there exists a nonnegative integer s such that

$$(n+1)^{2/3}A^2(\alpha_n^*)=a_s+\mathrm{O}(n^{-1/3}), \tag{4.3}$$

where a_s is the sth negative zero of the Airy function. Since a_s is negative, $A(\alpha_n^*)$ must tend to the negative imaginary axis (see Fig. 11), and the limit point of α_n^* must belong to the curve C_3 in Fig. 1. On the other hand, if $(n+1)^{2/3}A^2(\alpha_n^*)$ is unbounded, then it can be shown that the expansion in (3.25) reduces to that in (2.26) or (2.27), depending on whether $\alpha_n^*\in R_1$ or $\alpha_n^*\in R_2$; see the remark following Theorem B. From expansion (2.27), it is clear that α_n^* does not belong to R_2.

Hence, $\alpha_n^* \in R_1$ and (2.26) gives

$$e^{(n+1)f(\zeta_+,\alpha_n^*)}[C_0^{(+)} + O(n^{-1})] - e^{(n+1)f(\zeta_-,\alpha_n^*)}[C_0^{(-)} + O(n^{-1})] = 0,$$

or equivalently

$$(n+1)[f(\zeta_+,\alpha_n^*) - f(\zeta_-,\alpha_n^*)] + \log\frac{C_0^{(+)}}{C_0^{(-)}} + O(n^{-1}) = 2k\pi i, \tag{4.4}$$

where k is an integer. The last equation in turn yields

$$\mathrm{Re}\{f(\zeta_+,\alpha_n^*) - f(\zeta_-,\alpha_n^*)\} = O(n^{-1}), \quad \text{as } n \to \infty; \tag{4.5}$$

that is, α_n^* tends to the curve C_3 in Fig. 1. We have thus proved the following result.

Lemma 6. *The zeros of the normalized Generalized Bessel polynomial $y_n(\alpha/(n+1);a)$ tend to the curve C_3 in Fig. 1, as $n \to \infty$.*

This result has already been given by de Bruin et al. [4, p. 8]. However, it should be pointed out that their result holds uniformly only on compact subsets of $\mathbb{C}\backslash\{\alpha\colon \alpha = i(1+\tau),\ \tau > 0\}$, whereas our result holds uniformly in the whole upper half-plane. (See the statement following (1.16) in Section 1 and the qualifying phrase in Theorem 4.2 of [1].)

To obtain a more accurate formula than (4.3) for the zeros α_n^* of $y_n(\alpha/(n+1);a)$ near $\alpha = i$, we note that from (3.4) and (2.10), we have

$$A^3(\alpha) = -\tfrac{3}{4}\varphi(\alpha) = (1+i)(\alpha - i)^{3/2} + O\{(\alpha - i)^2\}.$$

Hence

$$A^2(\alpha) = (1+i)^{2/3}(\alpha - i) + O\{(\alpha - i)^{3/2}\}. \tag{4.6}$$

Inserting this into (4.3) gives

$$(\alpha_n^* - i) + O\{(\alpha_n^* - i)^{3/2}\} = (1+i)^{-2/3}\frac{a_s}{(n+1)^{2/3}} + O\left(\frac{1}{n+1}\right),$$

from which it follows that

$$\alpha_n^* = i + \frac{1}{2}(\sqrt{3} - i)\frac{a_s}{(n+1)^{2/3}} + O\left(\frac{1}{n+1}\right). \tag{4.7}$$

We conclude this paper by presenting a derivation of an asymptotic formula for the unique (negative) real zero of $y_n(\alpha/(n+1);a)$, a formula which was first (incorrectly) conjectured by Luke [7, p. 194]. To this end, we return to (2.28) and obtain

$$y_n(\alpha/(n+1);a) = -\frac{2^{a-1}}{\pi i}n!\left(\frac{2\alpha}{n+1}\right)^n e^{(n+1)/\alpha}$$
$$\times\left\{\tfrac{1}{2}e^{(n+1)f_-(\zeta_+,\alpha)}\left[\frac{g(\zeta_+)}{\sqrt{2}(-f_-''(\zeta_+,\alpha))^{1/2}}\sqrt{\frac{\pi}{n+1}} + O(n^{-3/2})\right]\right.$$

$$+\tfrac{1}{2}\,\mathrm{e}^{(n+1)f_+(\zeta_+,\alpha)}\left[\frac{g(\zeta_+)}{\sqrt{2}(-f_+''(\zeta_+,\alpha))^{1/2}}\sqrt{\frac{\pi}{n+1}}+\mathrm{O}(n^{-3/2})\right]$$

$$-\mathrm{e}^{(n+1)f(\zeta_-,\alpha)}\left[\frac{g(\zeta_-)}{\sqrt{2}(-f''(\zeta_-,\alpha))^{1/2}}\sqrt{\frac{\pi}{n+1}}+\mathrm{O}(n^{-3/2})\right]\Bigg\}. \tag{4.8}$$

Let $\bar{\alpha}_n(a)$ denote the negative real zero of $z_n(\alpha/(n+1);a)$. For convenience, we shall simply write $\bar{\alpha}$ for $\bar{\alpha}_n(a)$. Since $f_-''(\zeta_+,\alpha)=f_+''(\zeta_-,\alpha)$, replacing α by $\bar{\alpha}$ in (4.8) yields

$$\frac{1}{2}(\mathrm{e}^{(n+1)f_-(\zeta_+,\bar{\alpha})}+\mathrm{e}^{(n+1)f_+(\zeta_+,\bar{\alpha})})\frac{g(\zeta_+)}{\sqrt{2}(-f''(\zeta_+,\bar{\alpha}))^{1/2}}\sqrt{\frac{\pi}{n+1}}$$

$$\approx \mathrm{e}^{(n+1)f(\zeta_-,\bar{\alpha})}\frac{g(\zeta_-)}{\sqrt{2}(-f''(\zeta_-,\bar{\alpha}))^{1/2}}\sqrt{\frac{\pi}{n+1}}.$$

In view of (2.30) and (2.31), the last asymptotic equality can be written as

$$\exp\left\{(n+1)\left[\frac{\bar{\alpha}-\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}}-\log(2\bar{\alpha}^2-2\bar{\alpha}\sqrt{\bar{\alpha}^2+1})\right]\right\}$$

$$\times\cos[(n+1)\pi][1+(-\bar{\alpha}+\sqrt{\bar{\alpha}^2+1})]^{2-a}\left\{2\,\frac{1+(-\bar{\alpha}+\sqrt{\bar{\alpha}^2+1})^2}{[1-(-\bar{\alpha}+\sqrt{\bar{\alpha}^2+1})^2]^2}\right\}^{-1/2}$$

$$\approx\exp\left\{(n+1)\left[\frac{\bar{\alpha}+\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}}-\log(-2\bar{\alpha}\sqrt{\bar{\alpha}^2+1}-2\bar{\alpha}^2)\right]\right\}$$

$$\times[1+(-\bar{\alpha}-\sqrt{\bar{\alpha}^2+1})^2]^{2-a}\left\{2\frac{1+(-\bar{\alpha}-\sqrt{\bar{\alpha}^2+1})^2}{[1-(-\bar{\alpha}-\sqrt{\bar{\alpha}^2+1})^2]^2}\right\}^{-1/2}.$$

Taking logarithm on both sides leads to

$$(n+1)\left\{\frac{2\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}}+\log\frac{\sqrt{\bar{\alpha}^2+1}-\bar{\alpha}}{\sqrt{\bar{\alpha}^2+1}+\bar{\alpha}}\right\}+(2-a)\log\frac{1-\bar{\alpha}-\sqrt{\bar{\alpha}^2+1}}{1-\bar{\alpha}+\sqrt{\bar{\alpha}^2+1}}$$

$$+\frac{1}{2}\log\frac{\bar{\alpha}^2+1-\bar{\alpha}\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}^2+1+\bar{\alpha}\sqrt{\bar{\alpha}^2+1}}+\log\frac{\bar{\alpha}+\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}-\sqrt{\bar{\alpha}^2+1}}\approx-\log[\cos(n+1)\pi]. \tag{4.9}$$

Note that the quantities inside the first three logarithms in (4.9) are all real, and that the right-hand side vanishes when n is an odd integer. Thus, taking real parts, we have

$$\left\{\frac{2\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}}+\log\frac{\sqrt{\bar{\alpha}^2+1}-\bar{\alpha}}{\sqrt{\bar{\alpha}^2+1}+\bar{\alpha}}\right\}+\frac{1}{n+1}\left\{(2-a)\log\frac{1-\bar{\alpha}-\sqrt{\bar{\alpha}^2+1}}{1-\bar{\alpha}+\sqrt{\bar{\alpha}^2+1}}\right.$$

$$\left.+\frac{1}{2}\log\frac{\bar{\alpha}^2+1-\bar{\alpha}\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}^2+1+\bar{\alpha}\sqrt{\bar{\alpha}^2+1}}+\log\left|\frac{\bar{\alpha}+\sqrt{\bar{\alpha}^2+1}}{\bar{\alpha}-\sqrt{\bar{\alpha}^2+1}}\right|\right\}\approx 0. \tag{4.10}$$

To derive the result given by de Bruin et al. [4, p. 9], we rewrite (4.10) in such a way that the resulting equation is expressed in terms of $1/\bar{\alpha}$. Furthermore, we set

$$\frac{1}{\bar{\alpha}}=\frac{1}{\bar{\alpha}_n(a)}=\hat{r}+w(n), \tag{4.11}$$

where $\hat{r}$ is a negative real number and $w(n)=o(1)$ as $n\to\infty$. In terms of $\hat{r}$ and w, (4.10) gives

$$\begin{aligned}&\left\{\left[-2\sqrt{1+\hat{r}^2}-\frac{2\hat{r}}{\sqrt{1+\hat{r}^2}}w\right]+\left[\log\frac{\sqrt{1+\hat{r}^2}+1}{\sqrt{1+\hat{r}^2}-1}-\frac{2}{\hat{r}\sqrt{1+\hat{r}^2}}w\right]\right\}\\&\quad+\frac{1}{n+1}\left\{(2-a)\log\frac{\sqrt{1+\hat{r}^2}+\hat{r}-1}{\hat{r}-1-\sqrt{1+\hat{r}^2}}+\frac{1}{2}\log\frac{1+\hat{r}^2+\sqrt{1+\hat{r}^2}}{1+\hat{r}^2-\sqrt{1+\hat{r}^2}}+\log\frac{\sqrt{1+\hat{r}^2}-1}{\sqrt{1+\hat{r}^2}+1}\right\}\\&\quad+\mathrm{O}\left(\frac{w}{n+1}\right)=0.\end{aligned} \tag{4.12}$$

Comparing the O(1)-terms, we obtain

$$-2\sqrt{1+\hat{r}^2}+\log\frac{\sqrt{1+\hat{r}^2}+1}{\sqrt{1+\hat{r}^2}-1}=0,$$

i.e., $\hat{r}$ is a solution to the equation

$$\pm\hat{r}\,\mathrm{e}^{\sqrt{1+\hat{r}^2}}=1+\sqrt{1+\hat{r}^2}. \tag{4.13}$$

If we take '$-$' sign in (4.13), then this equation has only one negative root $\hat{r}=-0.662743\cdots$. Comparing terms in (4.12) of the next highest order of magnitude gives $w(n)=\mathrm{O}(1/n+1)$. More precisely, we have

$$\begin{aligned}w(n)=\frac{\hat{r}}{2\sqrt{1+\hat{r}^2}}&\left\{(2-a)\log\frac{-2\hat{r}}{(\hat{r}-1-\sqrt{1+\hat{r}^2})^2}+\frac{1}{2}\log\frac{(\sqrt{1+\hat{r}^2}+1)^2}{\hat{r}^2}\right.\\&\left.+\log\frac{\hat{r}^2}{(1+\sqrt{1+\hat{r}^2})^2}\right\}\frac{1}{n+1}+\mathrm{O}\left\{\frac{1}{(n+1)^2}\right\}.\end{aligned} \tag{4.14}$$

Each of the logarithmic terms in (4.14) can be simplified by using (4.13). For instance, the second log term in (4.14) is equal to $\sqrt{1+\hat{r}^2}$, and the third log term is equal to $-2\sqrt{1+\hat{r}^2}$. As to the first log term, we also have from (4.13)

$$(2-a)\log(-2\hat{r})=(2-a)[\log 2(1+\sqrt{1+\hat{r}^2})-\sqrt{1+\hat{r}^2}].$$

Thus, by letting

$$\bar{K}(\hat{r},a)\equiv\frac{1}{2}\left\{\frac{(2-a)\hat{r}}{\sqrt{1+\hat{r}^2}}\log(\hat{r}+\sqrt{1+\hat{r}^2})+(a-3)\hat{r}\right\}, \tag{4.15}$$

(4.14) can be written as

$$w(n)=\bar{K}(\hat{r},a)\frac{1}{n+1}+\mathrm{O}\left\{\frac{1}{(n+1)^2}\right\}. \tag{4.16}$$

In (4.15), we have made use of the identity

$$(\hat{r}+\sqrt{1+\hat{r}^2})(\hat{r}-1-\sqrt{1+\hat{r}^2})^2=2(1+\sqrt{1+\hat{r}^2}).$$

Inserting (4.16) into (4.11) yields

$$\frac{1}{\bar{\alpha}_n(a)}=\hat{r}+\bar{K}(\hat{r},a)\frac{1}{n+1}+\mathrm{O}\left\{\frac{1}{(n+1)^2}\right\}, \tag{4.17}$$

which is of course in complete agreement with the result (Theorem 7.3) in [4].

Let $\bar{z}_n(a)=\bar{\alpha}_n(a)/(n+1)$. Then $\bar{z}_n(a)$ is the unique negative real zero of $y_n(z;a)$. With $\hat{r} \risingdotseq -0.662743$, we have from (4.17)

$$\frac{2}{\bar{z}_n(a)} \risingdotseq 2n\hat{r}-1.006290a+1.349836+\mathrm{O}\left(\frac{1}{n}\right), \tag{4.18}$$

which is almost what Luke [7, p. 194] had incorrectly conjectured

$$\bar{z}_n(a)\sim -2[1.32548n+(a-1)-1/\pi]^{-1} \tag{4.19}$$

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